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PRODUCTS OF ABSOLUTE RETRACTS

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Let Φ be a class of graphs. $H \in \Phi$ is an *absolute retract* if for every isometric embedding h of H into a graph $G \in \Phi$ an edge-preserving map g from G to H exists such that $g \circ h$ is the identity map on H.

We show that the direct product of an absolute retract of n-chromatic graphs and an absolute retract of reflexive graphs yields a member of the former class.

This leads to a new characterization of absolute retracts of n-chromatic graphs.

Introduction

All graphs considered in this paper are finite, connected, undirected, and without multiple edges. For a graph G let $V(G)$ denote its vertex set and $E(G)$ its edge set. The edge connecting vertices u and v is denoted by (u, v). We agree upon $(u, v) = (v, u)$. We consider two classes: *reflexive graphs*, that is graphs in which every vertex is adjacent to itself, as well as *simple graphs*, that is graphs without loops at their vertices.

For any graphs G and H an *edge-preserving map* or homomorphism of G to H is a map f of $V(G)$ to $V(H)$ such that $f(g)$ is adjacent to $f(g')$ in H whenever g is adjacent to g' in G. A simple graph G is *n-chromatic* if there is a homomorphism onto K_n, the complete graph of n vertices, and n is the smallest such natural number. The integer n is also called the *chromatic number* $\chi(G)$ of G. An injective homomorphism is called an *embedding*. Only in reflexive graphs it is possible that an edge-preserving map identifies adjacent vertices.

H is a *subgraph* of G if $V(H) \subseteq V(G)$, and H has all edges inherited from G. H is a *retract* of G if H is a subgraph of G and there is a homomorphism $f: G \to H$ with $f(h) = h$ for all $h \in V(H)$, in this case f is called *retraction*.

If we denote by $d_G(g, g')$ the *distance* (= length $\ell(P_G)$ of a shortest path $P_G(g, g')$ *from* g to g' in G, it is easy to see that for any edge-preserving map f of G to H and any $g, g' \in V(G)$

$$d_H(f(g), f(g')) \leq d_G(g, g')$$

holds. Thus a subgraph H of a graph G cannot be a retract of G if there exists a "shortcut", i.e., a path of length $\ell < d_H(h, h')$ in G joining two vertices h, h' of H (see Fig. 1(a)).

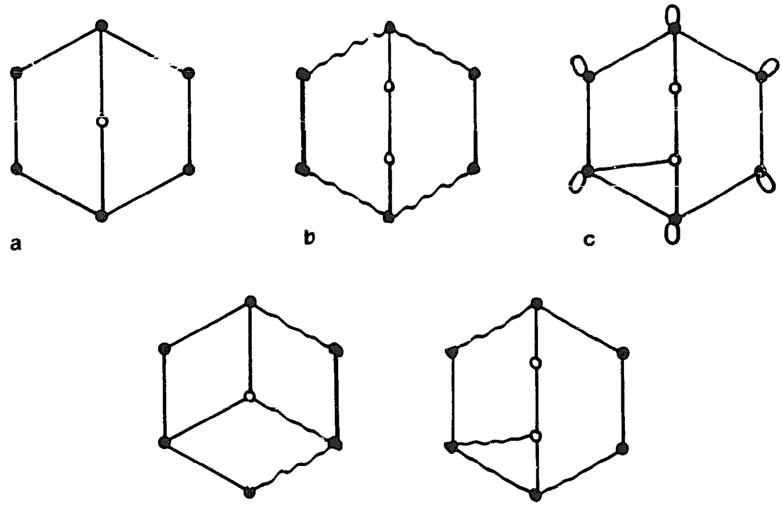

Fig. 1. The hexagon with shaded vertices is a retract of the graph in (b) and (c) but not of the graph in (a), (e) and (d).

If there is no shortcut we call a subgraph H of G *isometric*, i.e., distances between vertices are the same in H as in G:

$$d_H(h, h') = d_G(h, h'), \quad \text{for all } h, h' \in V(H).$$

Thus if a subgraph is a retract, then it is necessarily an isometric subgraph. Figure 1(d) shows that this condition is not sufficient. Reflexive graphs R which are retracts whenever this necessary condition is satisfied are called ***absolute retracts of reflexive graphs.***

It is easy to see that retractions on simple graphs preserve the chromatic number. Thus the hexagon in Fig. 1(e) cannot be a retract even though this is a retract in the reflexive case as Fig. 1(c) illustrates.

The ***absolute retracts of n-chromatic graphs*** are the graphs which are a retract whenever they are an isometric and isochromatic subgraph of an n-chromatic graph. We say, H is an *isochromatic* subgraph of G if the chromatic numbers of H and G are equal.

Hell [3, 4, 6] gave an early characterization of absolute retracts of bipartite graphs. These are also described by Bandelt, Dählmann and Schütte [1]. For n-chromatic graphs a recursive characterization was given by Poguntke and the author [15].

Nowakowski and Rival [11, 12] and Bandelt and the author [2] characterized the absolute retracts of reflexive graphs, see also Hell and Rival [7]. In [2], [7] and [12] absolute retracts of reflexive graphs are shown to be retracts of products of path.

In [1, 3, 6], especially [4] our main theorem is provided for the special case that all graphs are bipartite.

Moreover the theory of absolute retracts can be presented in a fairly general framework, see Misane [9], Jawhari [8], Jawhari, Misane and Pouzet [17]. Other classes of absolute retracts were investigated. Retracts of planar graphs are considered by Hell [5], such of partially ordered sets by Misane [9], Nevermann and Rival [10] or Quilliot [16], and for directed graphs, see Misane [9], Quilliot [16].

In this paper we shall consider both, absolute retracts of reflexive graphs and absolute retracts of n-chromatic graphs, and point out that the direct product of one of the first and one of the second class yields a member of the latter. This combination leads to another characterization of absolute retracts of n-chromatic graphs. They are just the retracts of a product of K_n and reflexive paths.

Basic results

By AR_n we denote the class of all absolute retracts of n-chromatic graphs, i.e., $G \in \mathrm{AR}_n$ if and only if:

(i) G is n-chromatic and simple, and
(ii) whenever G is an isometric subgraph of a simple graph G' which is also n-chromatic, then there is a retraction of G' onto G.

By ARR we denote the class of all absolute retracts of reflexive graphs, i.e., $G \in \mathrm{ARR}$ if and only if there is a retraction of the reflexive graph G' onto G whenever G is an isometric subgraph of G'.

The *diameter* of a graph G, diam G, is the maximum distance in G. Any vertex v which is at diameter distance from some other vertex is called a *diametrical vertex*.

$N_G(u)$ is called the *neighbourhood* of the vertex u in G and consists of all *neighbours* of u, i.e., all vertices of G which are adjacent to u. Note that $N_G(u)$ includes the vertex u if G is a reflexive graph, however if G is simple u does not belong to $N_G(u)$.

A vertex v of G satisfying $N_G(v) \subseteq N_G(w)$ for some vertex $w \in V(G)$, $v \neq w$, is called *embeddable* into w, equivalently, the vertex-deleted subgraph $G - v$ is a retract of G.

The following theorem is proved in Bandelt–Pesch [2] (see also Pesch–Poguntke [15]).

Theorem 1. *$G \in$ ARR if and only if every diametrical vertex v of G is embeddable and $G - v$ is a member of* ARR.

It is easy to see that in an absolute retract with diameter at least 3, each diametrical vertex is embeddable, irrespective of whether G belongs to ARR or AR_n. Moreover, every absolute retract with diameter 2 has at least 2 diametrical vertices which are embeddable. For details see [2, 13, 14, 15].

If the diameter of G is at most 2, then G is a member of ARR if and only if there exists a vertex z in G which is adjacent to all vertices of G.

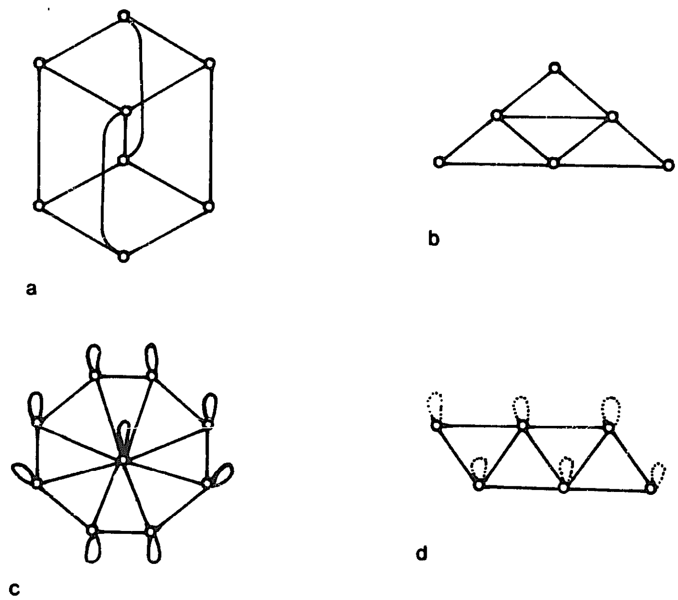

Fig. 2.

We also want to mention that any retract H of $G \in$ ARR also belongs to ARR and any retract H of $G \in \mathrm{AR}_n$ belongs to AR_n, too. For a proof see [12, 15].

Each $G \in \mathrm{AR}_n$ is uniquely n-colorable, i.e., there is a unique homomorphism of G onto K_n (up to an automorphism of K_n). The reason is that every vertex v of G belongs to a complete graph K_n in G and G always contains an embeddable vertex. Thus induction on the number of vertices of G yields the assertion. The reader can find a proof in [15].

The graphs in Figs. 2(a) and 2(b) are absolute retracts belonging to AR_n for $n=2$ and $n=3$, respectively. The graph in Fig. 2(c) is an absolute retract of reflexive graphs and that one in Fig. 2(d) belongs to both, ARR and AR_n. Trivially, $K_n \in \mathrm{AR}_n$.

Two theorems

As mentioned at the beginning, we need a suitable product which connects the two different classes of absolute retracts.

For simple graphs as well as for reflexive graphs the *direct product*

$$G := \bigtimes_{i=1}^{m} G_i$$

of graphs G_i, $i=1, \ldots, m$, is defined as follows: its vertex set is the cartesian product $V(G_1) \times \cdots \times V(G_m)$ and two vertices $g_1 g_2 \cdots g_m$ and $g_1' g_2' \cdots g_m'$ are adjacent in G if and only if (g_i, g_i') is an edge in G_i for all $i=1, \ldots, m$.

$H^{(r)}$ is the reflexive graph with the underlying simple graph H.

For technical reason we introduce the product $\otimes$. Let G and H be graphs. Then the product $G \otimes H$ is defined by

$$V(G \otimes H) := V(G) \times V(H),$$
$$E(G \otimes H) := E(G \times H) \cup \{(gh, g'h) \mid (g, g') \in E(G) \text{ and } h \in V(H)\}.$$

The following remarks are easy to check.

(i) If G and H are finite, connected graphs with $|V(G)| \geqslant 2$, then $G \otimes H$ is also connected and finite.

(ii) If G and H are simple graphs with chromatic number of G $\chi(G) = n \geqslant 2$ and $\chi(H) = m \geqslant 1$, then $\chi(G \otimes H) = n$.

(iii) For any path P the product $K_n \otimes P$ is uniquely n-colorable.

(iv) $G \otimes H = G \times H^{(r)}$ for two simple graphs G and H.

Proposition. *Let G and H be two simple graphs with* $\operatorname{diam} G \geqslant 1$. *Then for all $gh, g'h' \in V(G \otimes H)$ the following holds*:

$$\max\{d_G(g, g'); d_H(h, h')\} \leqslant d_{G\otimes H}(gh, g'h') \leqslant \max\{d_G(g, g'); d_H(h, h')\} + 1.$$

Proof. Let gh and $g'h'$ be two vertices of $G \otimes H$. Let $P_G(g, g') := (g, g_1, g_2, \ldots, g_{k-1}, g')$ be a shortest path from g to g' in G and let $P_H(h, h') := (h, h_1, h_2, \ldots, h_{\ell-1}, h')$ be a shortest path which joins h and h' in H. Then

$$P_{G\otimes H}(gh, g'h') := \begin{cases} (gh, g_1h_1, \ldots, g_{\ell-1}h_{\ell-1}, g_\ell h', \ldots, g_{k-1}h', g'h') & \text{if } k \geqslant \ell \\ (gh, g_1h_1, \ldots, g_{k-1}h_{k-1}, g'h_k, g_{k-1}h_{k+1}, g'h_{k+2}, \ldots, g'h') & \\ \qquad \text{if } k < \ell \text{ and } k = \ell \pmod 2 & \\ (gh, g_1h_1, \ldots, g_{k-1}h_{k-1}, g'h_k, g_{k-1}h_{k+1}, g'h_{k+2}, \ldots, g_{k-1}h', g'h') & \\ \qquad \text{else} & \end{cases}$$

is a path of length $\ell(P_{G\otimes H}(gh, g'h')) \geqslant d_{G\otimes H}(gh, g'h')$ in $G \otimes H$. □

The above proposition immediately implies the following:

Remark. For two simple graphs G and H with $\operatorname{diam} G \geqslant 1$, $\max\{\operatorname{diam} G; \operatorname{diam} H\} \leqslant \operatorname{diam} G \otimes H \leqslant \max\{\operatorname{diam} G; \operatorname{diam} H\} + 1$.

Before we are going to prove the next theorem we need the following

Proposition. *Let G be a (connected, finite) n-chromatic graph which is uniquely n-colorable. Then $G \times H^{(r)}$ is uniquely n-colorable for any (connected, finite) simple graph H.*

Proof. This is easy to see by induction on the number of edges of H. If $|E(H)|=0$, then $H\cong K_1$ and $G\times H^{(r)}\cong G$, whence we are done.

If H contains an edge such that its deletion does not disconnect H then let (h',h'') be such an edge of H with pendant vertices $h',h''\in V(H)$, and let H' be that graph which results from H when we delete the edge (h',h'') without removing the vertices h' and h'' from H. Otherwise, i.e., if H is a tree, then let (h',h'') be any pendant edge of H, i.e., $N_H(h'')=\{h'\}$. Then let H' be that graph which results from H when we delete the edge (h',h'') and the vertex h'', without removing h' from H. Then by the induction hypothesis $G\times H'^{(r)}$ is uniquely n-colorable. We get an n-coloring as follows: if $c_G:G\to K_n$ is an n-coloring of G then $c':G\times H'^{(r)}\to K_n$ with $c'(gh):=c_G(g)$ for all vertices $gh\in V(G\times H'^{(r)})$ with $g\in V(G)$ and $h\in V(H)$ is an n-coloring of $G\times H'^{(r)}$.

The graph $G\times H^{(r)}$ results from $G\times H'^{(r)}$ only by addition of the edges $(gh',g'h'')$ (and vertices $g'h''$ for all $g'\in V(G)$) where $(g,g')\in E(G)$, and $(h',h'')\in E(H)$. Thus consider the map $c:G\times H^{(r)}\to K_n$ with $c(gh):=c_G(g)$. Since $g\neq g'$ for any additional edge $(gh',g'h'')$ of $G\times H^{(r)}$ which is not in $G\times H'^{(r)}$, it is $c(gh')=c'(gh')\neq c'(g'h'')=c(g'h'')$ if H is not a tree, otherwise $c(gh')=c_G(g)\neq c_G(g')=c(g'h'')$. Thus c is also an n-coloring of the vertex set of $G\times H^{(r)}$, obviously unique, because $G\times H'^{(r)}$ and G are uniquely colorable. □

The next theorem gives a characterization of absolute retracts of reflexive graphs and of absolute retracts of n-chromatic graphs. It combines both classes of absolute retracts.

Theorem 2. *For each $n\geq 2$, $G\in \mathrm{AR}_n$ and $H^{(r)}\in \mathrm{ARR}$ if and only if $G\times H^{(r)}\in \mathrm{AR}_n$.*

Proof. Let $G\otimes H\in \mathrm{AR}_n$. $G\in \mathrm{AR}_n$ follows immediately since G is a retract of $G\otimes H$. So we only have to show that $H^{(r)}$ is an absolute retract.

Case 1. $\operatorname{diam} G\otimes H\leq 2$

It is necessary and sufficient to find a vertex z in $H^{(r)}$ which is adjacent to all other vertices of $H^{(r)}$.

For $V(H):=\{h_1,h_2,\ldots,h_m\}$ and any fixed vertex $g\in V(G)$ all vertices $gh_1,\ldots,gh_m$ have the same color, because G and $G\otimes H$ are uniquely n-colorable.

Let us now add to the graph $G\otimes H$ a new vertex x which is adjacent to all vertices gh_i, $1\leq i\leq m$, and denote the resulting graph $(G\otimes H)'$.

Now $G\otimes H$ is an isometric and isochromatic subgraph of $(G\otimes H)'$ and there exists a retraction $r:(G\otimes H)'\to G\otimes H$ which maps the vertex x to a vertex $\bar{g}h_k$ with $k\in\{1,\ldots,m\}$ and $\bar{g}\in V(G)$. With the definition $z:=h_k\in V(H)$ it follows that $(z,h_j)\in E(H^{(r)})$, for all vertices h_j of the reflexive graph $H^{(r)}$, and $H^{(r)}$ is a member of ARR.

Case 2. $\operatorname{diam} G \otimes H > 2$

Induction on the number of vertices of G.

Case 2.1. $G \cong K_n$. In particular this yields $\operatorname{diam} H > \operatorname{diam} G$. Using that each diametrical vertex of $G \otimes H$ is embeddable, we want to show that also any diametrical vertex v of $H^{(r)}$ is embeddable, $G \otimes (H - v) \in \mathrm{AR}_n$, and hence $H^{(r)}$ satisfies Theorem 1.

Let $P_H(v, w) := (v, u_1, u_2, \ldots, u_{d-1}, u_d, w)$ be a path of length $\ell(P_H(v, w)) = d_H(v, w) = \operatorname{diam} H$ in the graph H. Then there exists for any two vertices $x, y \in V(G)$ a shortest path $P_{G\otimes H}(xv, yw)$ from the vertex xv to yw in $G \otimes H$ with length $\ell(P_{G\otimes H}(xv, yw))$, where

$$\operatorname{diam} G \otimes H \geqslant \ell(P_{G\otimes H}(xv, yw)) = d_{G\otimes H}(xv, yw),$$

$$d_{G\otimes H}(xv, yw) = \begin{cases} d_H(v, w), & \text{if } n \geqslant 3, \text{ or if } n = 2 \text{ and} \\ & d_G(x, y) = d_H(v, w) \pmod 2, \\ d_H(v, w) + 1, & \text{else.} \end{cases}$$

For each vertex $z \in N_G(y)$ and each $u \in N_{H^{(r)}}(w)$ (zu, yw) is an edge in $G \otimes H$. Since yw is a diametrical vertex of $G \otimes H$ (see the first proposition and the remark above), yw is embeddable in a vertex, say $y'w' \in V(G \otimes H)$. Thus $N_{H^{(r)}}(w) \subseteq N_{H^{(r)}}(w')$.

If $n \geqslant 3$ it is possible to embed the vertex yw nearer by at least 1 to the vertex xv because the vertex $\overline{yw}$ at distance $d_{G\otimes H}(xv, yw)$ from xv is embeddable, perhaps at least 1 nearer to xv in the absolute retract $(G \otimes H) - yw$, where yw is assumed to be embeddable in $\overline{yw}$. Successive application for the vertex-deleted subgraphs yields the extra condition for yw. If $n = 2$ the distance from the diametrical associate of yw to the vertex in which yw is embeddable is $\operatorname{diam} G \otimes H - 2 = d_H(v, w) - 1$, whence we conclude $w \neq w'$.

So $H^{(r)} - w$ is a retract of $H^{(r)}$. Introduction on the number of vertices of H yields $H^{(r)} - w \in \mathrm{ARR}$ because $G \otimes (H - w)$ is a retract of $G \otimes H$ and therefore $G \otimes (H - w) \in \mathrm{AR}_n$ holds, too. Since this holds for all diametrical vertices $w \in V(H)$, we conclude with Theorem 1 that $H^{(r)} \in \mathrm{ARR}$.

Case 2.2. $K_n \not\cong G$. In the graph G we can find a vertex x which is embeddable into a vertex $y \in V(G)$ because G is an absolute retract. Now every vertex xu of $G \otimes H$ is embeddable in $yu \in V(G \otimes H)$ for all $u \in V(H)$, therefore $G - x \otimes H$ is a retract of $G \otimes H$. Induction on the number of vertices of G yields $H^{(r)} \in \mathrm{ARR}$.

Suppose now that $G \in \mathrm{AR}_n$ and $H^{(r)} \in \mathrm{ARR}$. Let $G \otimes H$ be an isometric subgraph of an n-chromatic graph T. We have to show now that $G \otimes H$ is a retract of T, and find a retraction of T to $G \otimes H$ via retractions of graphs G_T to G and $(H^{(r)})_T$ to $H^{(r)}$ where G_T and $(H^{(r)})_T$ relate to T in some way.

We divide the proof in several steps.

(i) Construction of the graphs G_T and $(H^{(r)})_T$ from G and $H^{(r)}$. The vertex set of G_T consists of the vertices of G and the vertices of T without the vertices of $G \otimes H$, i.e.,

$$V(G_T) := V(G) \cup V(T) \setminus V(G \otimes H).$$

The edge set of G_T is

$$\begin{aligned} E(G_T) := E(G) \cup \{(g, t) \mid (gh, t) \in E(T) \text{ with } g \in V(G), \\ h \in V(H), gh \in V(G \otimes H), t \in V(T) \setminus V(G \otimes H)\} \\ \cup \{(t_1, t_2 \in E(T) \mid t_1, t_2 \in V(T) \setminus V(G \otimes H)\}. \end{aligned}$$

The definition of $(H^{(r)})_T$ is analogous to that of G_T.

$$V((H^{(r)})_T) := V(H) \cup V(T) \setminus V(G \otimes H),$$

and

$$\begin{aligned} E((H^{(r)})_T) := E(H^{(r)}) \cup \{(h, t) \mid (gf, t) \in E(T) \text{ with } g \in V(G), \\ h \in V(H), gh \in V(G \otimes H), t \in V(T) \setminus V(G \otimes H)\} \\ \cup \{(t_1, t_2) \in E(T) \mid t_1, t_2 \in V(T) \setminus V(G \otimes H)\}. \end{aligned}$$

All other vertices of T which are not adjacent to any vertex of $G \otimes H$ have the same adjacencies as in T.

(ii) G is isochromatic in G_T, $\chi(G) = \chi(G_T)$. Since $G \otimes H$ is uniquely n-colorable we may color $G \otimes H$ first for each n-coloring of T. For a given n-coloring $c_G : V(G) \to \{1, \ldots, n\}$ of G, there exists also an n-coloring $c_{G \otimes H} : V(G \otimes H) \to \{1, \ldots, n\}$ of $G \otimes H$ with $c_{G \otimes H}(gh) = c_G(g)$ for each $gh \in V(G \otimes H)$. Therefore G_T is also n-chromatic if $\chi(T) = n$. Every vertex $t \in V(T) \setminus V(G \otimes H)$ may receive the same color in G_T as in T.

(iii) G is isometric in G_T. We have to show now that G is an isometric subgraph of G_T. Let us assume that there is a pair of vertices $g, g' \in V(G)$ with $d_{G_T}(g, g') < d_G(g, g')$. Then there are two vertices $h, h' \in V(H)$ with

$$d_T(gh, g'h') \leq d_{G_T}(g, g') < d_G(g, g') \leq d_{G \otimes H}(gh, g'h'),$$

which contradicts to the isometry of $G \otimes H$ in T.

Analogously we can see that $H^{(r)}$ is isometric in $(H^{(r)})_T$.

(iv) Construction of the retraction $f : T \to G \otimes H$. Since both G and $H^{(r)}$ are absolute retracts, there are retractions $r_G : G_T \to G$ and $r_H : (H^{(r)})_T \to H^{(r)}$. For $t \in V(T) \setminus V(G \otimes H)$ with $r_G(t) = g_t$ and $r_H(t) = h_t$, respectively, we define the function $f : T \to G \otimes H$ by

$$f(x) := \begin{cases} g_t h_t, & \text{if } x = t, \\ x, & \text{else, i.e., if } x \in V(G \otimes H). \end{cases}$$

Obviously, f is a retraction. Hence $G \otimes H \in \mathrm{AR}_n$ which finishes the proof. $\square$

An application of the above theorem yields another characterization of absolute retracts of n-chromatic graphs because each path $P_q^{(r)}$ of length q is an absolute retract of reflexive graphs by Theorem 1 (see also [2, 7, 11, 12]).

Theorem 3. *For each $n \in \mathbb{N}$, $G \in \mathrm{AR}_n$ if and only if there are natural numbers s and q such that G is a retract of*

$$K_n \times \mathop{\times}_{i=1}^{s} P_q^{(r)}.$$

Proof. It is convenient to define

$$R_q^s(n) := K_n \times \mathop{\times}_{i=1}^{s} P_q^{(r)}.$$

In [14] the author provides an isometric embedding of an arbitrary n-colored graph G into the product $R_q^s(n)$ with $q := \operatorname{diam} G$ and $s := |V(G)|$ as follows: Let $V(G) := \{v_1, \dots, v_s\}$. We define the embedding of G in $R_q^s(n)$ while we attach to each $v_i \in V(G)$ an $s+1$ – tuple $\bar{v}_i := (c(v), d_G(v_i, v_1), \dots, d_G(v_i, v_s))$, where c is an n-coloring of G. It is easy to check that the above mapping describes an isochromatic embedding. For the isometry and some more details see [14]. If $G \in \mathrm{AR}_n$ is isometrically embedded into $R_q^s(n)$, then G is a retract of this product.

Now let us assume that G is a retract of $R_q^s(n)$ for some $q, s \in \mathbb{N}_0$. First we point out that $R_q^s(n) \in \mathrm{AR}_n$ for any $q, s \in \mathbb{N}_0$. Here we define $R_q^0 := K_n$. We proceed by induction on s. If $s = 0$, then $R_q^0(n) \in \mathrm{AR}_n$ and for $s = 1$ Theorem 2 yields the assertion $R_q^1(n) = K_n \times P_q^{(r)} \in \mathrm{AR}_n$ for all $q \in \mathbb{N}_0$. For each $s \geqslant 2$ we have $R_q^s(n) = R_q^{s-1}(n) \times P_q^{(r)}$. Again Theorem 2 together with the induction hypothesis $R_q^{s-1}(n) \in \mathrm{AR}_n$ yields $R_q^s(n) \in \mathrm{AR}_n$, for all $q \in \mathbb{N}_0$. Since each retract of an absolute retract is also one, we have $G \in \mathrm{AR}_n$ which proves the theorem. □

Figure 3 gives an example to Theorem 3. The graph G is a retract of

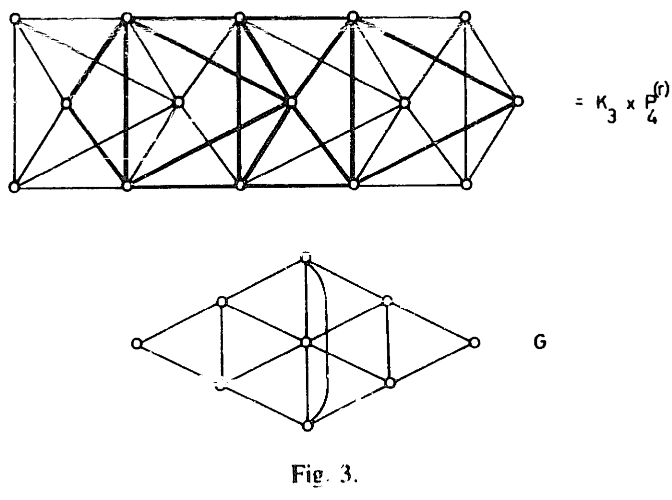

Fig. 3.

$K_3 \times P_4^{(r)} \in \mathrm{AR}_n$ hence $G \in \mathrm{AR}_n$. The heavily drawn lines of $K_3 \times P_4^{(r)}$ yield the isometric embedding of G.

Proposition. *If R_G is a retract of an n-chromatic graph G and $R_H^{(r)}$ is a retract of a reflexive graph $H^{(r)}$, then the product $R_G \times R_H^{(r)}$ is a retract of the product $G \times H^{(r)}$ which is n-chromatic.*

Proof. Let $r_G: G \to R_G$ and $r_H: H^{(r)} \to R_H^{(r)}$ be two retractions. If we define a map $r: G \times H^{(r)} \to R_G \times R_H^{(r)}$ by $r(gh) := r_G(g) r_H(h)$ for all $gh \in V(G \times H^{(r)})$, then it is easy to see that r is a retraction. □

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